The Arithmetic-Geometric Mean

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An Introduction

This project, as its title suggests, focuses on the *arithmetic-geometric mean.* Often abbreviated as AGM, it is an iteration on two numbers using the arithmetic and geometric means. Generally referred to as just the mean or average, the sum of two numbers divided by two is actually the *arithmetic* mean. The *geometric* mean is the square root of the two numbers multiplied together. As David Cox said in his paper on the subject [2]:

> *'This [the AGM] first appeared in a paper of Lagrange, but it was Gauss who really discovered the amazing depth of this subject.'*

He continues to say that the majority of Gauss' work was published after his death (as *Werke*, see [3]). Although they will not be covered in this project, Gauss also worked on a complex AGM.

French mathematician Legendre developed many of the ideas on *elliptic integrals.* Their name arose because one such integral gives the arc length of an ellipse — an example of this is demonstrated in Section 2. These types of integrals were the first types that mathematicians could not solve analytically, hence their relationship with the AGM.

An important application of the AGM, which is the application we cover, is its use in computing π. One algorithm in particular will be covered, but there are many more detailed in Borwein and Borwien [1]. To further this application, there will be a more detailed look at computing π using a program written by the author.

1. The Arithmetic-Geometric Mean

Given two non-negative numbers a and b, the *arithmetic mean* and *geometric mean* are given respectively by

$$\frac{a+b}{2} \qquad \text{and} \qquad \sqrt{ab}. \tag{1}$$

For example, if we have $a = 8$ and $b = 2$, then the arithmetic mean is $\frac{8+2}{2} = 5$ and the geometric mean is $\sqrt{8 \times 2} = 4$. The arithmetic-geometric mean is related to these two means and much of the underlying theory for this section is in Cox [2].

Let $a_0 = a$ and $b_0 = b$, with $a \geqslant b > 0$, and define the recursion:

$$a_{n+1} = \frac{a_n + b_n}{2} \qquad \text{and} \qquad b_{n+1} = \sqrt{a_n b_n}. \tag{2}$$

Note that b_{n+1} should always be the positive square root. This just means that both a_n and b_n are sequences and it is straightforward to see that a_1 and b_1 are the arithmetic and geometric means of a and b, that a_2 and b_2 are the respective means of a_1 and b_1, and so on. As we prove later, both sequences a_n and b_n have a common limit, which is the AGM.

Definition 1.1. The *arithmetic-geometric mean* M is defined by

$$M(a,b) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n,$$

where a_n and b_n are given in (2) and $n \in \{0,1,\dots\}$.

Looking at an example may assist in understanding the concept. The initial values, of $a = 25$ and $b = 4$, for the example below have been chosen at random.

Example 1.2. Calculate $M(25,4)$.

Solution. The following table shows the first four iterations of a_n and b_n as defined above in Definition 1.1.

i	a_i	b_i
0	25	4
1	14.5	10
2	12.25	12.04159
3	12.14579	12.14535
4	12.14557	12.14557

Intuitively, both sequences look to be approaching a similar value as n increases. This value, which is the AGM, appears to be around 12.146. (Note that no rounding has been applied when displaying the above table.)

The two sequences in Example 1.2 do actually converge to a common limit (rather than just appearing to). It can be proven that the sequences a_n and b_n converge for any initial values for a and b, not just for $a = 25$ and $b = 4$, and this leads to our first theorem.

Theorem 1.3. *For any $a \geqslant b \geqslant 0$, the arithmetic-geometric mean $M(a,b)$ exists.*

Proof. The standard inequality for arithmetic and geometric means states that

$$\frac{a+b}{2} \geqslant \sqrt{ab}.$$

This tells us that $a_i \geqslant b_i$ for any $i \in \{0,1,\dots\}$. Letting $i = n$ and $i = n+1$ gives

$$a_n \geqslant b_n \qquad \text{and} \qquad a_{n+1} \geqslant b_{n+1}.$$

Since $a_n \geqslant b_n$, this produces

$$a_n \geqslant \frac{a_n + b_n}{2} \qquad \text{and} \qquad \sqrt{a_n b_n} \geqslant b_n.$$

Using (2), we can write,

$$\begin{aligned} a_n \geqslant \frac{a_n+b_n}{2} = a_{n+1} \geqslant b_{n+1} = \sqrt{a_n b_n} \geqslant b_n \\ \implies \qquad a_n \geqslant a_{n+1} \geqslant b_{n+1} \geqslant b_n. \end{aligned} \tag{3}$$

This leads to

$$a \geqslant a_1 \geqslant \cdots \geqslant a_n \geqslant a_{n+1} \geqslant b_{n+1} \geqslant b_n \geqslant \cdots \geqslant b_1 \geqslant b.$$

Therefore, both a_n and b_n are bounded so, the limits $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist. The next step is to show that these limits are equal. So, from (3), we have

$$-b_{n+1} \leqslant -b_n.$$

Adding a_{n+1} produces

$$a_{n+1} - b_{n+1} \leqslant a_{n+1} - b_n$$

which, using (3), leads to

$$\begin{aligned} a_{n+1} - b_{n+1} &\leqslant \frac{a_n + b_n}{2} - b_n \\ \Longrightarrow \qquad a_{n+1} - b_{n+1} &\leqslant \frac{1}{2}\left(a_n - b_n\right). \end{aligned}$$

Then, we can write

$$a_n - b_n \leqslant \frac{1}{2}\left(a_{n-1} - b_{n-1}\right)$$

and iterate to give

$$\frac{1}{2}\left(a_{n-1} - b_{n-1}\right) \leqslant \frac{1}{2^2}\left(a_{n-2} - b_{n-2}\right) \leqslant \cdots \leqslant \frac{1}{2^n}\left(a_0 - b_0\right).$$

Therefore,

$$a_n - b_n \leqslant \frac{1}{2^n}\left(a_0 - b_0\right).$$

Now, take the limit as $n \to \infty$

$$\lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n \leqslant 0,$$

and since $a_n \geqslant b_n$ for any n,

$$\lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = 0.$$

This proves that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ and, therefore, that $M(a,b)$ exists. □

Therefore, we know the AGM exists. Now, it is useful to look at how quickly the AGM iteration converges.

Corollary 1.4. *The arithmetic-geometric mean converges quadratically.*

Proof. Using (2),

$$\begin{aligned} a_{n+1} - b_{n+1} &= \frac{a_n + b_n}{2} - \sqrt{a_n b_n} \\ &= \frac{1}{2}\left(\sqrt{a_n} - \sqrt{b_n}\right)^2 \\ &= \frac{1}{2}\left(\frac{a_n - b_n}{\sqrt{a_n} + \sqrt{b_n}}\right)^2, \end{aligned}$$

and since $b_0 \leqslant b_n \leqslant a_n$,

$$a_{n+1} - b_{n+1} \leqslant \frac{1}{2}\left(\frac{a_n - b_n}{2\sqrt{b_0}}\right)^2.$$

Then,

$$a_{n+1} - b_{n+1} \leqslant \frac{1}{8b_0}(a_n - b_n)^2$$

and therefore, the AGM converges quadratically. □

There are four basic properties of the AGM included. The first two properties are trivial and the second two are important for later sections.

Proposition 1.5. *Given two real numbers $a > b > 0$:*

(a) $M(a,a) = a$,
(b) $M(a,0) = 0$,
(c) $M(a,b) = M(a_1,b_1) = M(a_2,b_2) = \cdots$,
(d) $M(\lambda a, \lambda b) = \lambda M(a,b)$ *for any* $\lambda > 0 \in \mathbb{R}$.

The proofs of these properties are omitted, but are hopefully clear when looking at Definition 1.1.

2. Elliptic Integrals

The relationship between the arithmetic-geometric mean and elliptic integrals was of great interest to nineteenth century mathematicians [2]. Elliptic integrals originated from attempts to calculate the arc length of ellipses and the next result is perhaps the most important to this project.

Theorem 2.1. *For any two real numbers $a \geqslant b > 0$,*

$$I(a,b) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}} = \frac{\pi}{2}\frac{1}{M(a,b)}.$$

Proof. The first step is to prove that

$$I(a,b) = I(a_1,b_1).$$

Let us introduce a new variable θ such that

$$\sin\phi = \frac{2a\sin\theta}{(a+b)+(a-b)\sin^2\theta}. \tag{4}$$

The limits on the integral remain unchanged, since $0 \leqslant \theta \leqslant \frac{\pi}{2}$ corresponds to $0 \leqslant \phi \leqslant \frac{\pi}{2}$. This substitution was first used by Gauss [3], and we need to show that

$$\cos\phi = \frac{2\cos\theta\sqrt{a_1{}^2\cos^2\theta + b_1{}^2\sin^2\theta}}{(a+b)+(a-b)\sin^2\theta}, \tag{5}$$

where a_1 and b_1 are the arithmetic and geometric means of a and b as defined in (2). By squaring (4), we have an expression for $\cos^2\phi$:

$$\cos^2\phi = 1 - \sin^2\phi = 1 - \frac{4a^2\sin^2\theta}{\left[(a+b)+(a-b)\sin^2\theta\right]^2}.$$

To make the calculations simpler, let us set

$$\cos^2\phi = \frac{N}{\left[(a+b)+(a-b)\sin^2\theta\right]^2},$$

where

$$N = \left[(a+b) + (a-b)\sin^2\theta\right]^2 - 4a^2\sin^2\theta.$$

Rearranging (2), with $a = a_0$ and $b = b_0$, gives

$$4ab = 4{b_1}^2 \qquad \text{and} \qquad a^2 - 2ab + b^2 = 4({a_1}^2 - {b_1}^2). \tag{6}$$

We can expand N to give

$$\begin{aligned}
N &= (a+b)^2 - 2(a^2+b^2)(1-\cos^2\theta) + (a-b)^2\left(1-\cos^2\theta\right)^2\\
&= (a+b)^2 - 2(a^2+b^2) + 2\cos^2\theta(a^2+b^2)\\
&\quad + (a-b)^2\left(1 - 2\cos^2\theta + \cos^4\theta\right)\\
&= (a+b)^2 - 2(a^2+b^2) + 2(a^2+b^2)\cos^2\theta\\
&\quad + (a-b)^2 - 2(a-b)^2\cos^2\theta + (a-b)^2\cos^4\theta\\
&= 2\left[(a^2+b^2) - (a-b)^2\right]\cos^2\theta + (a-b)^2\cos^4\theta\\
&= \cos^2\theta\left[4ab + (a^2-2ab+b^2)\cos^2\theta\right]\\
&= \cos^2\theta\left[4ab + (a^2-2ab+b^2)(1-\sin^2\theta)\right]
\end{aligned}$$

and, by using (6),

$$\begin{aligned}
N &= 4\cos^2\theta\left[{b_1}^2 + ({a_1}^2 - {b_1}^2)(1-\sin^2\theta)\right]\\
&= 4\cos^2\theta\left[{a_1}^2(1-\sin^2\theta) + {b_1}^2\sin^2\theta\right].
\end{aligned}$$

Therefore,

$$N = 4\cos^2\theta\left[{a_1}^2\cos^2\theta + {b_1}^2\sin^2\theta\right],$$

which is what we required in (5).

Now we need to show that

$$\sqrt{a^2\cos^2\phi + b^2\sin^2\phi} = a\,\frac{(a+b) - (a-b)\sin^2\theta}{(a+b) + (a-b)\sin^2\theta}. \tag{7}$$

Denote the square of the left-hand side of (7) by A, that is

$$A = a^2\cos^2\phi + b^2\sin^2\phi = a^2 - (a^2-b^2)\sin^2\phi.$$

Substituting $\sin\phi$ as in (4) produces

$$\begin{aligned}
A &= a^2 - (a^2-b^2)\left(\frac{2a\sin\theta}{(a+b)+(a-b)\sin^2\theta}\right)^2\\
&= a^2\left(1 - \frac{4(a^2-b^2)\sin^2\theta}{\left[(a+b)+(a-b)\sin^2\theta\right]^2}\right)\\
&= a^2\left(\frac{(a+b)^2 - 2(a^2-b^2)\sin^2\theta + (a-b)^2\sin^4\theta}{\left[(a+b)+(a-b)\sin^2\theta\right]^2}\right)\\
&= a^2\left(\frac{\left[(a+b)-(a-b)\sin^2\theta\right]^2}{\left[(a+b)+(a-b)\sin^2\theta\right]^2}\right).
\end{aligned}$$

This can be rewritten as

$$A = \left(a\frac{(a+b)-(a-b)\sin^2\theta}{(a+b)+(a-b)\sin^2\theta} \right)^2,$$

which agrees with (7).

Differentiating (4) explicitly gives

$$\begin{aligned}\cos\phi\, d\phi &= \frac{2a\cos\theta\left[(a+b)+(a-b)\sin^2\theta\right]-4a(a-b)\sin^2\theta\cos\theta}{\left[(a+b)+(a-b)\sin^2\theta\right]^2}\, d\theta \\ &= \frac{2a\cos\theta\left[(a+b)+(a-b)\sin^2\theta-2(a-b)\sin^2\theta\right]}{\left[(a+b)+(a-b)\sin^2\theta\right]^2}\, d\theta\,,\end{aligned}$$

which simplifies to

$$\cos\phi\, d\phi = \frac{2a\cos\theta\left[(a+b)-(a-b)\sin^2\theta\right]}{\left[(a+b)+(a-b)\sin^2\theta\right]^2}\, d\theta\,.$$

Substituting $\cos\phi$ as in (5) gives

$$\frac{2\cos\theta\sqrt{{a_1}^2\cos^2\theta+{b_1}^2\sin^2\theta}}{(a+b)+(a-b)\sin^2\theta}\, d\phi = \frac{2a\cos\theta\left[(a+b)-(a-b)\sin^2\theta\right]}{\left[(a+b)+(a-b)\sin^2\theta\right]^2}\, d\theta\,.$$

This simplifies to

$$\sqrt{{a_1}^2\cos^2\theta+{b_1}^2\sin^2\theta}\, d\phi = a\frac{(a+b)-(a-b)\sin^2\theta}{(a+b)+(a-b)\sin^2\theta}\, d\theta\,,$$

which, from (7), gives

$$\frac{1}{\sqrt{a^2\cos^2\phi+b^2\sin^2\phi}}\, d\phi = \frac{1}{\sqrt{{a_1}^2\cos^2\theta+{b_1}^2\sin^2\theta}}\, d\theta\,.$$

Therefore, the integral can be rewritten as

$$I(a,b) = \int_0^{\pi/2}\frac{d\theta}{\sqrt{{a_1}^2\cos^2\theta+{b_1}^2\sin^2\theta}},$$

which proves that $I(a,b) = I(a_1,b_1)$. Iterating gives

$$I(a,b) = I(a_1,b_1) = I(a_2,b_2) = \cdots$$

such that

$$I(a,b) = \lim_{n\to\infty} I(a_n,b_n) = I(\mu,\mu),$$

where $\mu = M(a,b)$. Then

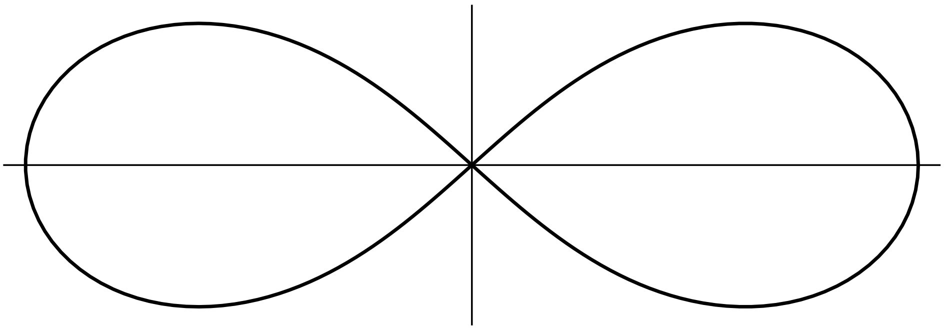
$$I(\mu,\mu) = \int_0^{\pi/2}\frac{1}{\mu}\, d\theta = \frac{\pi}{2\mu}.$$

Therefore,

$$I(a,b) = \frac{\pi}{2}\frac{1}{M(a,b)}$$

as required. □

This theorem helps us to solve many integrals, and one such integral arose from attempts to calculate the arc length of $r^2 = \cos 2\theta$. This curve is known as a *lemniscate* and it is plotted for $0 \leqslant \theta \leqslant 2\pi$.



Example 2.2. Calculate the arc length L of $r^2 = \cos 2\theta$.

Solution. The standard formula for the arc length of a curve $y(x)$ is given by

$$\int \sqrt{1 + \left(\tfrac{dy}{dx}\right)^2}\, dx\,. \tag{8}$$

However, our curve is defined in polar coordinates so we need to make two substitutions:

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta.$$

Differentiating with respect to θ gives

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad \text{and} \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta - r\cos\theta. \tag{9}$$

This allows us to write $\frac{dy}{dx}$ in terms of r and θ which is

$$\frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = \frac{\frac{dr}{d\theta}\sin\theta - r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}.$$

The radicand in (8) can then be rewritten in terms of r and θ:

$$\begin{aligned} 1 + \left(\frac{dy}{dx}\right)^2 &= \frac{\left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)^2 - \left(\frac{dr}{d\theta}\sin\theta - r\cos\theta\right)^2}{\left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)^2} \\ &= \frac{\left(\frac{dr}{d\theta}\right)^2 + r^2}{\left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)^2}. \end{aligned}$$

The integral in (8) is now

$$\int \sqrt{1 + \left(\tfrac{dy}{dx}\right)^2}\, dx = \int \frac{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}\, dx\,,$$

and using $\frac{dx}{d\theta}$ in (9) produces

$$\int \sqrt{1 + \left(\tfrac{dy}{dx}\right)^2}\, dx = \int \sqrt{r^2 + \left(\tfrac{dr}{d\theta}\right)^2}\, d\theta\,.$$

We are calculating the arc length of $r^2 = \cos 2\theta$ between 0 and 2π. We need $\frac{dr}{d\theta}$, and differentiating explicitly with respect to θ gives

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\theta}\left[r^2\right] &= \frac{\mathrm{d}}{\mathrm{d}\theta}\left[\cos 2\theta\right] \\ \implies \qquad 2r\frac{dr}{d\theta} &= -2\sin 2\theta. \end{aligned}$$

Squaring and substituting for r^2 produces

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{\sin^2 2\theta}{\cos 2\theta}.$$

This enables us to write

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \cos 2\theta + \frac{\sin^2 2\theta}{\cos 2\theta} = \frac{\cos^2 2\theta + \sin^2 2\theta}{\cos 2\theta},$$

which simplifies to

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \frac{1}{\cos 2\theta}.$$

We require the length of the curve between 0 and 2π but, if we look at the plot of the lemniscate, it is clear that the curve repeats four times in this interval. Therefore, we change the limits to 0 and $\pi/2$ and multiply the integral by 4. Then, the arc length L is given by

$$L = 4\int_0^{\pi/2} \sqrt{r^2 + \left(\tfrac{dr}{d\theta}\right)^2}\, d\theta = 4\int_0^{\pi/2} \frac{1}{\sqrt{\cos 2\theta}}\, d\theta\,. \tag{10}$$

Now, introduce a new variable t, such that

$$\cos 2\theta = \cos^2 t. \tag{11}$$

Differentiating (11) explicitly gives

$$\begin{aligned} -2\sin 2\theta \;\, d\theta &= -2\cos t \sin t \;\, dt \\ \implies \qquad d\theta &= \frac{\cos t \sin t}{\sin 2\theta}\, dt\,. \end{aligned}$$

Note that $0 \leqslant \theta \leqslant \pi/2$ corresponds to $0 \leqslant t \leqslant \pi/2$, so the limits remain unchanged. Continuing from (10) and (11) gives

$$L = 4\int_0^{\pi/2} \frac{1}{\sqrt{\cos^2 t}}\, d\theta\,,$$

and substituting for $d\theta$ produces

$$L = 4\int_0^{\pi/2} \frac{\sin t}{\sin 2\theta}\, dt\,.$$

From (11), we can say

$$\cos^2 2\theta = \cos^4 t,$$

and, therefore, that

$$\sin^2 2\theta = 1 - \cos^4 t = (1 - \cos^2 t)(1 + \cos^2 t) = \sin^2 t\,(1 + \cos^2 t).$$

Then,

$$L = 4\int_0^{\pi/2} \frac{\sin t}{\sin t\sqrt{1+\cos^2 t}}\,dt$$
$$= 4\int_0^{\pi/2} \frac{dt}{\sqrt{2\cos^2 t+\sin^2 t}}.$$

This integral is just $I\left(\sqrt{2},1\right)$. Therefore

$$L = \frac{2\pi}{M\left(\sqrt{2},1\right)}$$

is the arc length of $r^2 = \cos 2\theta$. For those interested, this evaluates to approximately 5.24412.

There are two types of elliptic integral that we need. The integral K below is the *complete elliptic integral of the first kind* and the integral E is the *complete elliptic integral of the second kind.* These integrals are complete because their limits are 0 and $\pi/2$. Be aware that some authors (such as Cox [2]) use F rather than K. Also, note that incomplete versions of both do exist and there is also an elliptic integral of the third kind, but it is not covered by this project.

Definition 2.3. For any $k \in [0,1)$, define K by

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

And, for any $k \in [0,1]$ define E by

$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\,d\theta\,.$$

Interestingly, K has a solution in terms of the AGM. As can be seen below, this is because the integral I in Theorem 2.1 is just a modified form of the complete integral of the first kind (specifically, where $a = 1$ and $b = \sqrt{1-k^2}$). Unfortunately, there is not such a nice relation for E, the second kind.

Proposition 2.4. *The integral K has the following solution in terms of the arithmetic-geometric mean M:*

$$K(k) = \frac{\pi}{2}\frac{1}{M(1+k,1-k)}.$$

Proof. By writing 1 as $\cos^2\theta + \sin^2\theta$, the integral K can be given as

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{\cos^2\theta+(1-k^2)\sin^2\theta}}.$$

This is just $I\big(1,\sqrt{1-k^2}\big)$, with I as in Theorem 2.1. Therefore,

$$K(k) = \frac{\pi}{2}\frac{1}{M(1+k,1-k)}.$$

The AGM can be rewritten since $M\big(1+k,1-k\big) = M\big(1,\sqrt{1-k^2}\big)$, by Proposition 1.5(c). □

It is useful to redefine the complete integrals of the first and second kind in terms of a different parameter $k' = \sqrt{1-k^2}$. These are known as complementary integrals and are denoted with a prime.

Definition 2.5. The *complementary integrals* K' and E' are defined by

$$K'(k) = K\big(\sqrt{1-k^2}\big) = K(k')$$

and

$$E'(k) = E\big(\sqrt{1-k^2}\big) = E(k'),$$

where $k' = \sqrt{1-k^2}$.

Sometimes, the parameter k is referred to as the *modulus* and k' as the *complementary modulus* [1]. There are some basic properties of K and E.

Proposition 2.6.

(a) $K(0) = \dfrac{\pi}{2}$, (b) $E(0) = \dfrac{\pi}{2}$, (c) $E(1) = 1$.

Proof.

(a) Using Proposition 2.4 produces

$$K(0) = \frac{\pi}{2}\frac{1}{M(1,1)} = \frac{\pi}{2}.$$

(b) Using Definition 2.3 with $k = 0$ gives

$$E(0) = \int_0^{\pi/2} \sqrt{1}\; d\theta = \frac{\pi}{2}.$$

(c) Now with $k = 1$ gives

$$E(1) = \int_0^{\pi/2} \sqrt{1-\sin^2\theta}\; d\theta = \int_0^{\pi/2} \cos\theta\; d\theta = \sin\frac{\pi}{2} = 1. \qquad \square$$

The integrals K and E can be related by the differential of K (why this is useful becomes clear in Theorem 4.1).

Proposition 2.7. *The differential of K with respect to k, denoted by $\dot{K}$, is:*

$$\dot{K} = \frac{E - k'^2 K}{k k'^2}$$

where $K = K(k)$ and $E = E(k)$.

Proof. The easiest proof of this uses the series expansions for K and E. Part of this involves calculating integrals of arbitrary powers of sine. So, let

$$S_n = \int_0^{\pi/2} \sin^n\theta\, d\theta = \int_0^{\pi/2} \sin^{n-1}\theta\, \sin\theta\; d\theta\,,$$

where n is a non-negative integer. Using integration by parts, we can then write

$$S_n = \Big[-\sin^{n-1}\theta\, \cos\theta \Big]_0^{\pi/2} + \int_0^{\pi/2} (n-1)\sin^{n-2}\theta\, \cos^2\theta\; d\theta\,.$$

Since $\cos\frac{\pi}{2} = 0 = \sin 0$, the first term evaluates to zero. Then $\cos^2\theta$ can be rewritten to produce

$$\begin{aligned} S_n &= \int_0^{\pi/2} (n-1)\sin^{n-2}\theta \left(1-\sin^2\theta\right) d\theta \\ &= (n-1)\left[\int_0^{\pi/2} \sin^{n-2}\theta \, d\theta - \int_0^{\pi/2} \sin^n\theta \, d\theta\right] \\ &= (n-1)\big[S_{n-2} - S_n\big]. \end{aligned}$$

Rearranging gives

$$S_n = \left(\frac{n-1}{n}\right) S_{n-2}, \tag{12}$$

where $S_0 = \int_0^{\pi/2} d\theta = \frac{\pi}{2}$ and $S_1 = \int_0^{\pi/2} \sin\theta \, d\theta = 1$.

First, we will calculate the series expansion for K. So, Definition 2.3 states that

$$K(k) = \int_0^{\pi/2} \left(1 - k^2\sin^2\theta\right)^{-1/2} d\theta\,.$$

This integral can be expanded using the binomial theorem (which states that $(1+x)^n = \sum_{r=0}^{\infty} \frac{n(n-1)\cdots(n-r+1)}{r!}x^r$). Therefore,

$$\begin{aligned} K(k) &= \int_0^{\pi/2} \Bigg[1 + \frac{\left(-\frac{1}{2}\right)}{1!}\left(-k^2\sin^2\theta\right)^1 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-k^2\sin^2\theta\right)^2 \\ &\qquad + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(-k^2\sin^2\theta\right)^3 + \cdots\Bigg] d\theta \\ &= \int_0^{\pi/2} \Bigg[1 + \frac{1}{2\times 1!}\left(k^2\sin^2\theta\right)^1 + \frac{1\times 3}{2^2\times 2!}\left(k^2\sin^2\theta\right)^2 \\ &\qquad + \frac{1\times 3\times 5}{2^3\times 3!}\left(k^2\sin^2\theta\right)^3 + \cdots\Bigg] d\theta \\ &= \int_0^{\pi/2} \left[\sum_{r=0}^{\infty} \frac{(2r-1)!!}{2^r\, r!}\left(k^2\sin^2\theta\right)^r\right] d\theta\,, \end{aligned}$$

where $(2r-1)!! = 1\times 3\times 5\times\cdots\times(2r-1)$ is a double factorial. Taking the summation out of the integral leaves

$$K(k) = \sum_{r=0}^{\infty} \frac{(2r-1)!!}{2^r\, r!} k^{2r} \int_0^{\pi/2} \sin^{2r}\theta \, d\theta = \sum_{r=0}^{\infty} \frac{(2r-1)!!}{2^r\, r!} k^{2r}\, S_{2r}.$$

Using (12), we know that $S_0 = \frac{\pi}{2}$, and we can see that $S_2 = \frac{1}{2}S_0 = \frac{1}{2\times 1}\frac{\pi}{2}$. Also, $S_4 = \frac{3}{4}S_2 = \frac{3}{2\times 2}\frac{1}{2\times 1}\frac{\pi}{2}$ and $S_6 = \frac{5}{6}S_4 = \frac{5}{2\times 3}\frac{3}{2\times 2}\frac{1}{2\times 1}\frac{\pi}{2}$. So, we can calculate any S_{2r} using the following formula:

$$S_{2r} = \frac{(2r-1)\times(2r-3)\times\cdots\times 1}{(2\times r)\times(2\times r-1)\times\cdots\times(2\times 1)}\frac{\pi}{2} = \frac{(2r-1)!!}{2^r\, r!}\frac{\pi}{2}. \tag{13}$$

Therefore,

$$K(k) = \sum_{r=0}^{\infty} \frac{(2r-1)!!}{2^r\, r!} k^{2r} \, \frac{(2r-1)!!}{2^r\, r!} \frac{\pi}{2}$$

$$\implies \quad K(k) = \frac{\pi}{2} \sum_{r=0}^{\infty} \left[\frac{(2r-1)!!}{2^r\, r!} \right]^2 k^{2r}. \tag{14}$$

We also need $\dot{K} = \frac{dK}{dk}$. We do not differentiate K as in defined in Definition 2.3 since it is much simpler to differentiate the series above. So, differentiating (14) with respect to k gives

$$\dot{K}(k) = \frac{\pi}{2} \sum_{r=0}^{\infty} \left[\frac{(2r-1)!!}{2^r\, r!} \right]^2 2r\, k^{2r-1}. \tag{15}$$

Using a similar method, we need to calculate the series expansion for E. Therefore, Definition 2.3 states that

$$E(k) = \int_0^{\pi/2} \left(1 - k^2 \sin^2\theta\right)^{1/2} d\theta\,,$$

and using binomial expansion gives

$$E(k) = \int_0^{\pi/2} \Bigg[1 + \frac{\left(\frac{1}{2}\right)}{1!} \left(-k^2 \sin^2\theta\right)^1 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!} \left(-k^2 \sin^2\theta\right)^2$$
$$+ \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!} \left(-k^2 \sin^2\theta\right)^3 + \cdots \Bigg] d\theta\,.$$

Then, rearranging and introducing a double factorial produces

$$E(k) = \int_0^{\pi/2} \Bigg[1 - \frac{1}{2 \times 1!} \left(k^2 \sin^2\theta\right)^1 - \frac{1 \times 1}{2^2 \times 2!} \left(k^2 \sin^2\theta\right)^2$$
$$- \frac{1 \times 1 \times 3}{2^3 \times 3!} \left(k^2 \sin^2\theta\right)^3 - \cdots \Bigg] d\theta$$
$$= \int_0^{\pi/2} \left[1 - \sum_{r=1}^{\infty} \frac{(2r-3)!!}{2^r\, r!} \left(k^2 \sin^2\theta\right)^r \right] d\theta$$
$$= \int_0^{\pi/2} \left[1 - \sum_{r=1}^{\infty} \frac{(2r-1)!!}{2^r\, r!} \frac{1}{2r-1} \left(k^2 \sin^2\theta\right)^r \right] d\theta\,.$$

Note that $-1!! = 1$ (not -1) so $\frac{(2r-1)!!}{2r-1} = \frac{1 \times 3 \times \cdots \times (2r-3) \times (2r-1)}{2r-1} = (2r-3)!!$. This allows us to rewrite the summation in a way that is similar to the series for K. Then, rearranging gives

$$E(k) = \int_0^{\pi/2} 1\, d\theta - \sum_{r=1}^{\infty} \frac{(2r-1)!!}{2^r\, r!} \frac{k^{2r}}{2r-1} \int_0^{\pi/2} \sin^{2r}\theta\, d\theta$$
$$= \frac{\pi}{2} - \sum_{r=1}^{\infty} \frac{(2r-1)!!}{2^r\, r!} \frac{k^{2r}}{2r-1} S_{2r}.$$

Using (13) produces

$$E(k) = \frac{\pi}{2} - \sum_{r=1}^{\infty} \frac{(2r-1)!!}{2^r\, r!} \frac{k^{2r}}{2r-1} \frac{(2r-1)!!}{2^r\, r!} \frac{\pi}{2}$$

$$\implies \quad E(k) = \frac{\pi}{2}\left(1 - \sum_{r=1}^{\infty} \left[\frac{(2r-1)!!}{2^r\, r!}\right]^2 \frac{k^{2r}}{2r-1}\right). \tag{16}$$

We are proving that $\dot{K} = (E - k'^2 K)/kk'^2$, which can be rearranged to $kk'^2\dot{K} = E - k'^2 K$ and substituting for k' gives

$$(k - k^3)\dot{K} = E - (1 - k^2)K. \tag{17}$$

Denote the coefficient of k^{2n} in the left-hand side and right-hand side of the above equation by L and R respectively. Showing that $L = R$ for an arbitrary power of k proves the proposition. For ease of writing, $\mathcal{C}(k^a)$ denotes the coefficient of k^a. So, we have

$$L = \mathcal{C}(k^{2n}) \text{ in } (k\dot{K} - k^3\dot{K}) = \left[\mathcal{C}(k^{2n-1}) \text{ in } \dot{K}\right] - \left[\mathcal{C}(k^{2n-3}) \text{ in } \dot{K}\right].$$

Now, using (15), the coefficient of k^{2n-1} is when $r = n$ and the coefficient of k^{2n-3} is when $r = n - 1$. Therefore,

$$L = \frac{\pi}{2}\left[\frac{(2n-1)!!}{2^n\, n!}\right]^2 2n \; - \; \frac{\pi}{2}\left[\frac{(2n-3)!!}{2^{n-1}\,(n-1)!}\right]^2 2(n-1).$$

Using $(2n-3)!! = (2n-1)!!/(2n-1)$ and $(n-1)! = n!/n$ allows us to write:

$$\frac{(2n-3)!!}{2^{n-1}\,(n-1)!} = \frac{(2n-1)!!}{2^n\, n!} \frac{2n}{(2n-1)}.$$

Then,

$$\begin{aligned} L &= \frac{\pi}{2}\left[\frac{(2n-1)!!}{2^n\, n!}\right]^2 2n \; - \; \frac{\pi}{2}\left[\frac{(2n-1)!!}{2^n\, n!}\right]^2 \frac{(2n)^2}{(2n-1)^2} \\ &= \frac{\pi}{2}\left[\frac{(2n-1)!!}{2^n\, n!}\right]^2 \left[2n - \frac{(2n)^2\, 2(n-1)}{(2n-1)^2}\right], \end{aligned}$$

which leads to

$$L = \frac{\pi}{2}\left[\frac{(2n-1)!!}{2^n\, n!}\right]^2 \frac{2n}{(2n-1)^2}. \tag{18}$$

Similarly,

$$\begin{aligned} R &= \mathcal{C}(k^{2n}) \text{ in } (E - K + k^2 K) \\ &= \left[\mathcal{C}(k^{2n}) \text{ in } E\right] - \left[\mathcal{C}(k^{2n}) \text{ in } K\right] + \left[\mathcal{C}(k^{2n-2}) \text{ in } K\right]. \end{aligned}$$

Using (16), the coefficient of k^{2n} is when $r = n$. And using (14), the coefficient of k^{2n} and k^{2n-2} is when $r = n$ and $r = n - 1$ respectively. Therefore,

$$\begin{aligned} R &= -\frac{\pi}{2}\left[\frac{(2n-1)!!}{2^n\, n!}\right]^2 \frac{1}{2n-1} \; - \; \frac{\pi}{2}\left[\frac{(2n-1)!!}{2^n\, n!}\right]^2 \; + \; \frac{\pi}{2}\left[\frac{(2n-3)!!}{2^{n-1}\,(n-1)!}\right]^2 \\ &= \frac{\pi}{2}\left[\frac{(2n-1)!!}{2^n\, n!}\right]^2 \left[-\frac{1}{2n-1} - 1 + \frac{(2n)^2}{(2n-1)^2}\right]. \end{aligned}$$

Then rearranging gives

$$R = \frac{\pi}{2}\left[\frac{(2n-1)!!}{2^n\, n!}\right]^2 \frac{2n}{(2n-1)^2},$$

which agrees with (18). Therefore, this proves the proposition. □

3. The Gamma & Beta Functions

This section may appear out of place, but certain values of k in K and E have a solution which can be expressed in terms of the gamma function. It is defined below along with the beta function.

Definition 3.1. The *gamma* function is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt\,,$$

where $\Re(x) > 0$.

Definition 3.2. The *beta* function is defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt\,,$$

where $\Re(x), \Re(y) > 0$.

The following two theorems, and their proofs, are taken from Titchmarsh [5]. The first theorem relates the gamma function to itself (it allows us to show the common result $\Gamma(x) = (x-1)!$) and the second redefines the beta function in terms of gamma.

Theorem 3.3. *For any* $\Re(x) > 0$,

$$\Gamma(x+1) = x\,\Gamma(x).$$

Proof. From Definition 3.1, we have

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t}\, dt\,.$$

Integration by parts produces

$$\Gamma(x+1) = \Big[-t^x e^{-t}\Big]_0^\infty + \int_0^\infty x\, t^{x-1} e^{-t}\, dt\,.$$

The first term evaluates to zero and therefore,

$$\Gamma(x+1) = x\int_0^\infty t^{x-1} e^{-t}\, dt = x\,\Gamma(x),$$

as required. □

Theorem 3.4. *The* beta *function is also given by*

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$$

where $\Re(x), \Re(y) > 0$.

Proof. Using Definition 3.1, we can write

$$\begin{aligned}\Gamma(x)\,\Gamma(y) &= \int_0^\infty t^{x-1}e^{-t}\,dt \int_0^\infty s^{y-1}e^{-s}\,ds \\ &= \int_0^\infty \int_0^\infty e^{-t-s}t^{x-1}s^{y-1}\,dt\,ds\,.\end{aligned}$$

Now let $t = uv$ and let $s = u(1-v)$. Then $dt = u\,dv$ and $ds = -\,du$. The first integral's limits remain unchanged since $0 \leqslant t \leqslant \infty$ corresponds to $0 \leqslant u < \infty$. For the second, $0 \leqslant s < \infty$ corresponds to $1 \geqslant v \geqslant 0$ (since s is inversely proportional to v). Therefore,

$$\begin{aligned}\Gamma(x)\,\Gamma(y) &= \int_{u=0}^\infty \int_{v=1}^0 -e^{-u}(uv)^{x-1}\big(u[1-v]\big)^{y-1}u\,du\,dv \\ &= \int_0^\infty e^{-u}u^{x+y-1}\,du \int_0^1 v^{x-1}(1-v)^{y-1}\,dv\end{aligned}$$

Using Definition 3.1 and Definition 3.2 leads to

$$\Gamma(x)\,\Gamma(y) = \Gamma(x+y)\,B(x,y),$$

which proves the result. □

A useful relation between sine and the gamma function, which was first devised by Euler [4], is included below.

Theorem 3.5. *Euler's reflection formula is given by*

$$\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin \pi x},$$

where $0 < x < 1$.

This proof relies on some assumptions which are outside the scope of this project. Namely, that

$$\sin(\pi x) = \pi x \prod_{n=1}^\infty \left(1 - \frac{x^2}{n^2}\right), \tag{19}$$

$$\frac{1}{\Gamma(x)} = xe^{\gamma x} \prod_{n=1}^\infty \left(1 + \frac{x}{n}\right)e^{-x/n}, \tag{20}$$

where $\gamma \approx 0.5772156649$ is Euler's constant. The proof, and these assumptions, are covered in Havil [4].

Proof. Using (20),

$$\begin{aligned}\frac{1}{\Gamma(x)}\frac{1}{\Gamma(-x)} &= \left[xe^{\gamma x}\prod_{n=1}^\infty\left(1+\frac{x}{n}\right)e^{-x/n}\right]\left[-xe^{-\gamma x}\prod_{n=1}^\infty\left(1-\frac{x}{n}\right)e^{x/n}\right] \\ &= -x^2e^{\gamma x}e^{-\gamma x}\prod_{n=1}^\infty\left(1+\frac{x}{n}\right)\left(1-\frac{x}{n}\right)e^{-x/n}e^{x/n} \\ &= -x^2\prod_{n=1}^\infty\left(1-\frac{x^2}{n^2}\right).\end{aligned}$$

From Theorem 3.3, we can write $\Gamma(1-x) = -x\Gamma(-x)$. Then,

$$\frac{1}{\Gamma(x)}\frac{1}{\Gamma(1-x)} = x\prod_{n=1}^\infty\left(1-\frac{x^2}{n^2}\right).$$

Therefore, using (19) produces

$$\frac{1}{\Gamma(x)}\frac{1}{\Gamma(1-x)} = \frac{\sin \pi x}{\pi},$$

as required. □

For Theorem 4.4 in the next section, we need to calculate the gamma function at two specific values of x.

Proposition 3.6.

(a) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$,

(b) $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\,\pi$.

Proof. For (a), let $x = \frac{1}{2}$ in Theorem 3.5 which produces $\Gamma^2\left(\frac{1}{2}\right) = \pi$. For (b), the result follows directly by letting $x = \frac{1}{4}$. □

4. Elliptic Integrals Continued

Before we can approach an algorithm relating π and the AGM, we need to look further at elliptic integrals. More specifically, we are interested in how K and E relate to each other and how we can solve specific integrals using the gamma function. The following theorem appears in Borwein and Borwein [1].

Theorem 4.1. *For any* $k \in (0,1)$*:*

(a) $K(k) = \dfrac{1}{1+k}\, K\left(\dfrac{2\sqrt{k}}{1+k}\right)$,

(b) $K(k) = \dfrac{2}{1+k'}\, K\left(\dfrac{1-k'}{1+k'}\right)$,

(c) $E(k) = \dfrac{1+k}{2}\, E\left(\dfrac{2\sqrt{k}}{1+k}\right) + \dfrac{k'^2}{2}\, K(k)$,

(d) $E(k) = (1+k')\, E\left(\dfrac{1-k'}{1+k'}\right) - k'\, K(k)$.

Proof.

(a) Using Proposition 2.4 we can write

$$K(k) = I(1+k, 1-k),$$

and using Proposition 1.5(d),

$$K(k) = \frac{1}{1+k}\, I\left(1, \frac{1-k}{1+k}\right). \tag{21}$$

Note that $I(1+k, 1-k) = I\left(1, \sqrt{1-k^2}\right)$ since 1 and $\sqrt{1-k^2}$ are, respectively, the arithmetic and geometric means of $1+k$ and $1-k$. Therefore, we can write

$$K\left(\frac{2\sqrt{k}}{1+k}\right) = I\left(1, \sqrt{1-\left(\frac{2\sqrt{k}}{1+k}\right)^2}\right)$$

$$= I\left(1, \sqrt{\frac{(1+k)^2-4k}{(1+k)^2}}\right) = I\left(1, \sqrt{\frac{(1-k)^2}{(1+k)^2}}\right).$$

Now, multiplying by $\frac{1}{1+k}$ produces

$$\frac{1}{1+k}\,K\left(\frac{2\sqrt{k}}{1+k}\right) = \frac{1}{1+k}\,I\left(1, \frac{1-k}{1+k}\right). \tag{22}$$

Combining (21) with (22) proves the result.

(b) Similarly, we can write

$$K(k) = I\left(1, \sqrt{1-k^2}\right) = I(1, k'), \tag{23}$$

where $k' = \sqrt{1-k^2}$ as before. Then, using (21) produces

$$K\left(\frac{1-k'}{1+k'}\right) = p\,I(1,q)$$

where

$$p = \frac{1}{1+\frac{1-k'}{1+k'}} = \frac{1+k'}{2} \qquad \text{and} \qquad q = \frac{1-\frac{1-k'}{1+k'}}{1+\frac{1-k'}{1+k'}} = k'.$$

Now, dividing by p gives

$$\frac{2}{1+k'}\,K\left(\frac{1-k'}{1+k'}\right) = I(1,k'). \tag{24}$$

Combining (23) with (24) proves the result.

(c) Let $g(k)$ and its differential be as follows:

$$g = \frac{2\sqrt{k}}{1+k} \qquad \text{and} \qquad \dot{g} = \frac{1-k}{\sqrt{k}(1+k)^2}.$$

Then, from (a) we have

$$(1+k)\,K(k) = K(g),$$

which differentiates (with respect to k) to produce

$$K(k) + (1+k)\,\dot{K}(k) = \dot{g}\,\dot{K}(g)$$

$$\implies \quad K(k) + (1+k)\,\dot{K}(k) = \frac{1-k}{\sqrt{k}(1+k)^2}\,\dot{K}(g). \tag{25}$$

Rearranging Proposition 2.7 gives

$$E(k) = kk'^2\dot{K}(k) + k'^2K(k), \tag{26}$$

and with $k = g$,

$$E(g) = gg'^2\dot{K}(g) + g'^2K(g). \tag{27}$$

Note that

$$g'^2 = 1 - g^2 = 1 - \frac{4k}{(1+k)^2} = \left(\frac{1-k}{1+k}\right)^2.$$

Then, if we take (26) $-\frac{1+k}{2}$(27):

$$\begin{aligned} E(k) - \tfrac{1+k}{2}E(g) &= kk'^2\dot{K}(k) + k'^2K(k) \\ &\quad - \tfrac{1+k}{2}gg'^2\dot{K}(g) - \tfrac{1+k}{2}g'^2K(g) \\ &= kk'^2\dot{K}(k) + k'^2K(k) \\ &\quad - \sqrt{k}\left(\tfrac{1-k}{1+k}\right)^2\dot{K}(g) - \tfrac{(1-k)^2}{2(1+k)}K(g). \end{aligned}$$

Now, from (a), we can rewrite the following:

$$\tfrac{(1-k)^2}{2(1+k)}K(g) = \tfrac{(1-k)^2}{2}K(k).$$

Therefore,

$$\begin{aligned} E(k) - \tfrac{1+k}{2}E(g) &= kk'^2\dot{K}(k) + k'^2K(k) \\ &\quad - \sqrt{k}\left(\tfrac{1-k}{1+k}\right)^2\dot{K}(g) - \tfrac{(1-k)^2}{2}K(k), \end{aligned}$$

and using (25),

$$\begin{aligned} E(k) - \tfrac{1+k}{2}E(g) &= kk'^2\dot{K}(k) + k'^2K(k) - \tfrac{(1-k)^2}{2}K(k) \\ &\quad - \sqrt{k}\left(\tfrac{1-k}{1+k}\right)^2\tfrac{\sqrt{k}(1+k)^2}{1-k}\left[K(k) + (1+k)\dot{K}(k)\right] \\ &= kk'^2\dot{K}(k) + k'^2K(k) - \tfrac{(1-k)^2}{2}K(k) \\ &\quad - k(1-k)K(k) - k(1-k)(1+k)\dot{K}(k). \end{aligned}$$

Since $k(1-k)(1+k) = k(1-k^2) = kk'^2$, the first and the last term give a zero coefficient for $\dot{K}$. This leaves:

$$\begin{aligned} E(k) - \tfrac{1+k}{2}E(g) &= K(k)\left[k'^2 - \tfrac{(1-k)^2}{2} - k(1-k)\right] \\ &= K(k)\left[(1-k^2) - \tfrac{1-2k+k^2}{2} - k + k^2\right] \\ &= K(k)\left[\tfrac{1}{2} - \tfrac{k^2}{2}\right]. \end{aligned}$$

Finally, substituting for $g(k)$ and since $1 - k^2 = k'^2$, then

$$E(k) - \frac{1+k}{2}E\left(\frac{2\sqrt{k}}{1+k}\right) = \frac{k'^2}{2}K(k),$$

as required.

(d) Let $h = \frac{1-k'}{1+k'}$. (Incidentally, h is actually the inverse of g.) Then, substituting $k = h$ in (c) gives:

$$\begin{aligned} E(h) &= \frac{1+h}{2}\, E\left(\frac{2\sqrt{h}}{1+h}\right) + \frac{h'^2}{2}\, K(h) \\ &= \frac{1+\frac{1-k'}{1+k'}}{2}\, E\left(\frac{2\sqrt{\frac{1-k'}{1+k'}}}{1+\frac{1-k'}{1+k'}}\right) + \frac{h'^2}{2}\, K\left(\frac{1-k'}{1+k'}\right) \\ &= \frac{1}{1+k'}\, E\left(\sqrt{1-k'^2}\right) + \frac{h'^2}{2}\, K\left(\frac{1-k'}{1+k'}\right), \end{aligned}$$

where

$$\frac{h'^2}{2} = \frac{1-h^2}{2} = \frac{1}{2}\left[1 - \left(\frac{1-k'}{1+k'}\right)^2\right] = \frac{2k'}{(1+k')^2}.$$

Therefore,

$$E(h) = \frac{1}{1+k'}\, E(k) + \frac{2k'}{(1+k')^2}\, K\left(\frac{1-k'}{1+k'}\right),$$

and using (b) gives

$$\begin{aligned} E(h) &= \frac{1}{1+k'}\, E(k) + \frac{2k'}{(1+k')^2}\frac{1+k'}{2}\, K(k) \\ &= \frac{1}{1+k'}\, E(k) + \frac{k'}{1+k'}\, K(k). \end{aligned}$$

Multiplying through by $1 + k'$ and substituting for $h(k)$ produces

$$(1+k')\, E\left(\frac{1-k'}{1+k'}\right) = E(k) + k' K(k),$$

which is what we required. □

The integral I (defined in Theorem 2.1) and similar integral J (defined below) both relate to E and K. This is expected given that we saw in Proposition 2.4 how K has a solution in terms of the AGM.

Proposition 4.2. *If*

$$\begin{aligned} J(a,b) &= \int_0^{\pi/2} \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\; d\theta = a\, E'\left(\frac{b}{a}\right) \\ I(a,b) &= \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}} = \frac{1}{a}\, K'\left(\frac{b}{a}\right), \end{aligned}$$

then

$$2\, J(a_{n+1}, b_{n+1}) - J(a_n, b_n) = a_n b_n\, I(a_n, b_n).$$

Proof. First we need to show that $J(a,b) = a\,E'\left(\frac{b}{a}\right)$. So,

$$\begin{aligned} J(a,b) &= \int_0^{\pi/2} \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\;d\theta \\ &= a\int_0^{\pi/2} \sqrt{\cos^2\theta + \tfrac{b^2}{a^2}\sin^2\theta}\;d\theta \\ &= a\int_0^{\pi/2} \sqrt{1 + \left(1 - \tfrac{b^2}{a^2}\right)\sin^2\theta}\;d\theta\,. \end{aligned}$$

Using Definition 2.3 and then Definition 2.5, we can write

$$J(a,b) = a\,E\left(\sqrt{1 - \tfrac{b^2}{a^2}}\right) = a\,E'\left(\frac{b}{a}\right). \tag{28}$$

Similarly, we need to show that $I(a,b) = \frac{1}{a}\,K'\left(\frac{b}{a}\right)$. So,

$$\begin{aligned} I(a,b) &= \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}} \\ &= \frac{1}{a}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1 + \left(1 - \tfrac{b^2}{a^2}\right)\sin^2\theta}}. \end{aligned}$$

Therefore,

$$I(a,b) = \frac{1}{a}\,K\left(\sqrt{1 - \tfrac{b^2}{a^2}}\right) = \frac{1}{a}\,K'\left(\frac{b}{a}\right). \tag{29}$$

Let ${c_n}^2 = {a_n}^2 - {b_n}^2$. Then set $k_n = c_n/a_n$ and so

$${k_n}' = \sqrt{1 - \frac{{c_n}^2}{{b_n}^2}} = \sqrt{\frac{{a_n}^2 - ({a_n}^2 - {b_n}^2)}{{a_n}^2}} = \frac{b_n}{a_n}.$$

Now, from Theorem 4.1,

$$\begin{aligned} E(k_n) &= (1 + {k_n}')\,E\left(\frac{1 - {k_n}'}{1 + {k_n}'}\right) - {k_n}'K(k_n) \\ \Longrightarrow \qquad E\left(\frac{c_n}{a_n}\right) &= \left(1 + \frac{b_n}{a_n}\right) E\left(\frac{a_n - b_n}{a_n + b_n}\right) - \frac{b_n}{a_n}\,K\left(\frac{c_n}{a_n}\right). \end{aligned}$$

Multiplying through by a_n gives

$$a_n\,E\left(\frac{c_n}{a_n}\right) = (a_n + b_n)\,E\left(\frac{a_n - b_n}{a_n + b_n}\right) - b_n\,K\left(\frac{c_n}{a_n}\right).$$

Note that $\frac{a_n - b_n}{a_n + b_n} = \frac{c_{n+1}}{a_{n+1}}$ and that $a_n + b_n = 2a_{n+1}$. Therefore,

$$a_n\,E\left(\frac{c_n}{a_n}\right) = 2a_{n+1}\,E\left(\frac{c_{n+1}}{a_{n+1}}\right) - a_n b_n\,K\left(\frac{c_n}{a_n}\right).$$

Since $E(k') = E'(k)$ and $K(k') = K'(k)$, we can write

$$a_n\,E\left(\frac{c_n}{a_n}\right) = 2a_{n+1}\,E\left(\frac{c_{n+1}}{a_{n+1}}\right) - a_n b_n\,K\left(\frac{c_n}{a_n}\right).$$

Finally, using (28) and (29) produces

$$2\,J(a_{n+1}, b_{n+1}) - J(a_n, b_n) = a_n b_n\,I(a_n, b_n),$$

as required. □

This theorem gives a more direct relation for K and E (previously, we could only relate them as in Theorem 4.1).

Theorem 4.3. *For $a = 1$ and $b = k' \in (0,1]$,*

$$E(k) = \left(1 - \sum_{n=0}^{\infty} 2^{n-1} {c_n}^2\right) K(k),$$

where ${c_n}^2 = {a_n}^2 - {b_n}^2$.

Proof. Using (2),

$$\begin{aligned} {c_n}^2 = {a_n}^2 - {b_n}^2 &= -(a_n + b_n)^2 + 2{a_n}^2 + 2a_n b_n \\ &= -4{a_{n+1}}^2 + 2{a_n}^2 + 2a_n b_n, \end{aligned}$$

therefore,

$$a_n b_n = \frac{1}{2}\left({c_n}^2 + 4{a_{n+1}}^2 - 2{a_n}^2\right). \tag{30}$$

From Proposition 4.2,

$$2\,J(a_{n+1}, b_{n+1}) - J(a_n, b_n) = a_n b_n\, I(a_n, b_n),$$

and using (30) gives,

$$2\,J(a_{n+1}, b_{n+1}) - J(a_n, b_n) = \frac{1}{2}\left[{c_n}^2 + 4{a_{n+1}}^2 - 2{a_n}^2\right] I(a_n, b_n).$$

Collecting similar indexes of a and b produces

$$\begin{aligned} &2\left[J(a_{n+1}, b_{n+1}) - {a_{n+1}}^2\, I(a_n, b_n)\right] \\ &\qquad - \left[J(a_n, b_n) - {a_n}^2\, I(a_n, b_n)\right] = \frac{1}{2}{c_n}^2\, I(a_n, b_n), \end{aligned}$$

and multiplying by 2^n gives

$$\begin{aligned} &2^{n+1}\left[J(a_{n+1}, b_{n+1}) - {a_{n+1}}^2\, I(a_n, b_n)\right] \\ &\qquad -2^n\left[J(a_n, b_n) - {a_n}^2\, I(a_n, b_n)\right] = 2^{n-1}{c_n}^2\, I(a_n, b_n). \end{aligned} \tag{31}$$

Note that we can rewrite $I(a_n, b_n)$ as $I(a_0, b_0)$ by Theorem 2.1. Now, by summing the left-hand side of (31) from $n = 1$ to $n = \infty$:

$$\begin{aligned} &2\left[J(a_1, b_1) - {a_1}^2\, I(a_0, b_0)\right] - 1\left[J(a_0, b_0) - {a_0}^2\, I(a_0, b_0)\right] \\ &+ 4\left[J(a_2, b_2) - {a_2}^2\, I(a_0, b_0)\right] - 2\left[J(a_1, b_1) - {a_1}^2\, I(a_0, b_0)\right] \\ &+ 8\left[J(a_3, b_3) - {a_3}^2\, I(a_0, b_0)\right] - 4\left[J(a_2, b_2) - {a_2}^2\, I(a_0, b_0)\right] \\ &\vdots \\ &+ 2^n\left[J(a_n, b_n) - {a_n}^2\, I(a_0, b_0)\right] - 2^{n-1}\left[J(a_{n-1}, b_{n-1}) - {a_{n-1}}^2\, I(a_0, b_0)\right]. \end{aligned}$$

It can be seen that many terms cancel out leaving only:

$$-\left[J(a_0, b_0) - {a_0}^2\, I(a_0, b_0)\right]. \tag{32}$$

To justify this, let

$$\begin{aligned}\Delta_n &= 2^n \left[{a_n}^2 I(a_n, b_n) - J(a_n, b_n)\right]\\ &= 2^n \int_0^{\pi/2} \frac{{a_n}^2 - \left({a_n}^2 \cos^2\theta + {b_n}^2 \sin^2\theta\right)}{\sqrt{{a_n}^2 \cos^2\theta + {b_n}^2 \sin^2\theta}}\, d\theta\\ &= 2^n \int_0^{\pi/2} \frac{\left({a_n}^2 - {b_n}^2\right)\sin^2\theta}{\sqrt{{a_n}^2 \cos^2\theta + {b_n}^2 \sin^2\theta}}\, d\theta\\ &= 2^n\, {c_n}^2 \int_0^{\pi/2} \frac{\sin^2\theta}{\sqrt{{a_n}^2 \cos^2\theta + {b_n}^2 \sin^2\theta}}\, d\theta\,.\end{aligned}$$

Since $0 \leqslant \sin^2\theta \leqslant 1$, we can write $0 \leqslant \Delta_n \leqslant 2^n\, {c_n}^2\, I(a_n, b_n)$ and then observe that $\Delta_n \to 0$ as $n \to \infty$. Now, summing the right-hand side of (31) produces

$$\sum_{n=0}^{\infty} 2^{n-1} {c_n}^2\, I(a_0, b_0),$$

and equating with (32) gives

$$\begin{aligned}&\sum_{n=0}^{\infty} 2^{n-1} {c_n}^2\, I(a_0, b_0) = -J(a_0, b_0) + {a_0}^2\, I(a_0, b_0)\\ \Longrightarrow \qquad & J(a_0, b_0) = \left({a_0}^2 - \sum_{n=0}^{\infty} 2^{n-1} {c_n}^2\right) I(a_0, b_0).\end{aligned}$$

From Proposition 4.2, this can be rewritten as

$$a_0\, E'\left(\frac{b_0}{a_0}\right) = \left({a_0}^2 - \sum_{n=0}^{\infty} 2^{n-1} {c_n}^2\right) \frac{1}{a_0}\, K'\left(\frac{b_0}{a_0}\right),$$

and since $a_0 = a = 1$ and $b_0 = b = k'$,

$$E'(k') = \left(1 - \sum_{n=0}^{\infty} 2^{n-1} {c_n}^2\right) K'(k').$$

Finally, using Proposition 2.4 gives

$$E(k) = \left(1 - \sum_{n=0}^{\infty} 2^{n-1} {c_n}^2\right) K(k).$$

as required. □

We need the value for K and E at $1/\sqrt{2}$ to enable us to calculate the result given in Corollary 4.5. These results both involve the gamma function discussed in the previous section.

Theorem 4.4.

(a) $K\left(\dfrac{1}{\sqrt{2}}\right) = \dfrac{\Gamma^2\left(\frac{1}{4}\right)}{4\sqrt{\pi}}$,

(b) $E\left(\dfrac{1}{\sqrt{2}}\right) = \dfrac{4\,\Gamma^2\left(\frac{3}{4}\right) + \Gamma^2\left(\frac{1}{4}\right)}{8\sqrt{\pi}}$.

Proof.

(a) From Definition 2.3 with $k = 1/\sqrt{2}$, we have

$$K\left(\frac{1}{\sqrt{2}}\right) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \frac{1}{2}\sin^2\theta}}.$$

Let $t = \sin\theta$ then $\frac{dt}{d\theta} = \cos\theta = \sqrt{1-t^2}$. Note the limits change since $0 \leqslant \theta \leqslant \frac{\pi}{2}$ corresponds to $0 \leqslant t \leqslant 1$. Therefore,

$$\begin{aligned} K\left(\frac{1}{\sqrt{2}}\right) &= \int_0^1 \frac{1}{\sqrt{1-\frac{1}{2}t^2}} \frac{dt}{\sqrt{1-t^2}} \\ &= \sqrt{2}\int_0^1 \frac{dt}{\sqrt{(2-t^2)(1-t^2)}}. \end{aligned}$$

Now let $t^2 = \dfrac{2x^2}{1+x^2}$, and differentiate explicitly:

$$\begin{aligned} t\,dt &= \frac{2x}{(1+x^2)^2}\,dx \\ \Longrightarrow \quad \frac{\sqrt{2}\,x}{\sqrt{1+x^2}}\,dt &= \frac{2x}{(1+x^2)^2}\,dx \\ \Longrightarrow \quad dt &= \frac{\sqrt{2}}{(1+x)^{3/2}}\,dx\,. \end{aligned}$$

Note that $0 \leqslant t \leqslant 1$ corresponds to $0 \leqslant x \leqslant 1$, so the limits remain unchanged. Then,

$$\begin{aligned} K\left(\frac{1}{\sqrt{2}}\right) &= \sqrt{2}\int_0^1 \frac{dt}{\sqrt{\left(1-\frac{2x^2}{1+x^2}\right)\left(2-\frac{2x^2}{1+x^2}\right)}} \\ &= \sqrt{2}\int_0^1 \frac{dt}{\sqrt{\left(\frac{1-x^2}{1+x^2}\right)\left(\frac{2}{1+x^2}\right)}} \\ &= \sqrt{2}\int_0^1 \frac{dt}{\sqrt{\frac{2}{(1+x^2)^2}}\sqrt{1-x^2}} \\ &= \int_0^1 \frac{dt}{(1+x^2)^{-1}\sqrt{1-x^2}}. \end{aligned}$$

And, substituting for dt produces

$$\begin{aligned} K\left(\frac{1}{\sqrt{2}}\right) &= \int_0^1 \frac{1}{(1+x^2)^{-1}\sqrt{1-x^2}} \frac{\sqrt{2}}{(1+x^2)^{3/2}}\,dx \\ &= \sqrt{2}\int_0^1 \frac{dx}{\sqrt{1+x^2}\sqrt{1-x^2}} \\ &= \sqrt{2}\int_0^1 \frac{dx}{\sqrt{1-x^4}}. \end{aligned} \tag{33}$$

Then, let $u = x^4$ so $\frac{du}{dx} = 4x^3 = 4u^{3/4}$. The limits remain unchanged since $0 \leqslant x \leqslant 1$ corresponds to $0 \leqslant u \leqslant 1$. Therefore,

$$\begin{aligned} K\left(\frac{1}{\sqrt{2}}\right) &= \sqrt{2}\int_0^1 \frac{1}{\sqrt{1-u}}\frac{1}{4u^{3/4}}\,du \\ &= \frac{\sqrt{2}}{4}\int_0^1 u^{-3/4}(1-u)^{-1/2}\,du\,. \end{aligned}$$

Definition 3.2, with $x = \frac{1}{4}$ and $y = \frac{1}{2}$, produces

$$K\left(\frac{1}{\sqrt{2}}\right) = \frac{\sqrt{2}}{4}\,B\!\left(\frac{1}{4},\frac{1}{2}\right),$$

and using Theorem 3.5 gives

$$K\left(\frac{1}{\sqrt{2}}\right) = \frac{\sqrt{2}}{4}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)}.$$

However, Proposition 3.6 tells us that

$$\Gamma\left(\frac{3}{4}\right) = \frac{\sqrt{2}\,\pi}{\Gamma\left(\frac{1}{4}\right)} \qquad \text{and} \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Therefore,

$$\begin{aligned} K\left(\frac{1}{\sqrt{2}}\right) &= \frac{\sqrt{2}}{4}\frac{\Gamma\left(\frac{1}{4}\right)\sqrt{\pi}}{\sqrt{2}\,\pi/\Gamma\left(\frac{1}{4}\right)} \\ \Longrightarrow \quad K\left(\frac{1}{\sqrt{2}}\right) &= \frac{\Gamma^2\left(\frac{1}{4}\right)}{4\sqrt{\pi}}. \end{aligned}$$

(b) From Definition 2.3 with $k = 1/\sqrt{2}$, we have

$$E\left(\frac{1}{\sqrt{2}}\right) = \int_0^{\pi/2}\sqrt{1-\tfrac{1}{2}\sin^2\theta}\;d\theta\,.$$

As before, let $t = \sin\theta$ then $\frac{dt}{d\theta} = \sqrt{1-t^2}$. The limits change in the same way. Then

$$\begin{aligned} E\left(\frac{1}{\sqrt{2}}\right) &= \int_0^1\sqrt{1-\tfrac{1}{2}t^2}\;d\theta \\ &= \int_0^1\frac{\sqrt{1-\frac{1}{2}t^2}}{\sqrt{1-t^2}}\;dt\,. \end{aligned}$$

Now let $t^2 = 1-u^2$ (which can be rearranged to $\frac{1}{2}+\frac{1}{2}u^2 = 1-\frac{1}{2}t^2$) and differentiate explicitly:

$$\begin{aligned} 2t\frac{dt}{du} &= -2u \\ \Longrightarrow \qquad \frac{dt}{du} &= \frac{-u}{\sqrt{1-u^2}}. \end{aligned}$$

Note that $0 \leqslant t \leqslant 1$ corresponds to $1 \geqslant u \geqslant 0$, so the limits do change. Therefore,

$$E\left(\frac{1}{\sqrt{2}}\right) = \int_{u=1}^{0} \frac{\sqrt{\frac{1}{2}+\frac{1}{2}u^2}}{u}\,dt = \int_{1}^{0} -\frac{\sqrt{\frac{1}{2}\left(1+u^2\right)}}{\sqrt{1-u^2}}\,du\,,$$

and multiplying by $1 = \sqrt{1+u^2}\big/\sqrt{1+u^2}$ gives

$$\begin{aligned} E\left(\frac{1}{\sqrt{2}}\right) &= \frac{1}{\sqrt{2}}\int_0^1 \frac{\sqrt{1+u^2}}{\sqrt{1-u^2}}\frac{\sqrt{1+u^2}}{\sqrt{1+u^2}}\,du \\ &= \frac{1}{\sqrt{2}}\int_0^1 \frac{1+u^2}{\sqrt{1-u^4}}\,du \\ &= \frac{1}{\sqrt{2}}\left[\int_0^1 \frac{1}{\sqrt{1-u^4}}\,du + \int_0^1 \frac{u^2}{\sqrt{1-u^4}}\,du\right]. \end{aligned}$$

The first integral in the above equation can be simplified using (33) to give

$$E\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}K\left(\frac{1}{\sqrt{2}}\right) + \int_0^1 \frac{u^2}{\sqrt{1-u^4}}\,du\right]. \tag{34}$$

We know the value of $K\left(1/\sqrt{2}\right)$, so we just need to solve the remaining integral. Therefore, let $x = u^4$ then $\frac{dx}{du} = 4x^{3/4}$ as in part (a) above. Now, we have

$$\begin{aligned} \int_0^1 \frac{u^2}{\sqrt{1-u^4}}\,du &= \int_0^1 \frac{\sqrt{x}}{\sqrt{1-x}}\,du \\ &= \int_0^1 \frac{\sqrt{x}}{\sqrt{1-x}}\frac{dx}{4x^{3/4}} \\ &= \frac{1}{4}\int_0^1 x^{-1/4}(1-x)^{-1/2}\,dx\,. \end{aligned}$$

By using Definition 3.2 and Theorem 3.5, this leads to

$$\int_0^1 \frac{u^2}{\sqrt{1-u^4}}\,du = \frac{1}{4}\,B\left(\frac{3}{4},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{4\,\Gamma\left(\frac{5}{4}\right)}.$$

From Theorem 3.3 and Proposition 3.6, we can write

$$\Gamma\left(\frac{5}{4}\right) = \frac{1}{4}\,\Gamma\left(\frac{1}{4}\right) = \frac{\sqrt{2}\,\pi}{4\,\Gamma\left(\frac{3}{4}\right)} \qquad \text{and} \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Therefore,

$$\int_0^1 \frac{u^2}{\sqrt{1-u^4}}\,du = \frac{\Gamma^2\left(\frac{3}{4}\right)\sqrt{\pi}}{\sqrt{2}\,\pi} = \frac{\Gamma^2\left(\frac{3}{4}\right)}{\sqrt{2}\sqrt{\pi}}.$$

Continuing from (34), and by using part (a), produces

$$\begin{aligned} E\left(\frac{1}{\sqrt{2}}\right) &= \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}K\left(\frac{1}{\sqrt{2}}\right) + \frac{\Gamma^2\left(\frac{3}{4}\right)}{\sqrt{2}\sqrt{\pi}}\right] \\ &= \frac{1}{2}\left[\frac{\Gamma^2\left(\frac{1}{4}\right)}{4\sqrt{\pi}} + \frac{\Gamma^2\left(\frac{3}{4}\right)}{\sqrt{\pi}}\right]. \end{aligned}$$

Simplifying gives

$$E\left(\frac{1}{\sqrt{2}}\right) = \frac{\Gamma^2\left(\frac{1}{4}\right) + 4\,\Gamma^2\left(\frac{3}{4}\right)}{8\sqrt{\pi}},$$

which proves the result. □

The two statements in the theorem above can be used to give a value for π in terms of elliptic integrals.

Corollary 4.5.

$$K\left(\frac{1}{\sqrt{2}}\right)\left[2\,E\left(\frac{1}{\sqrt{2}}\right) - K\left(\frac{1}{\sqrt{2}}\right)\right] = \frac{\pi}{2}$$

Proof. From Theorem 4.4, we know that

$$\begin{aligned} K\left(\tfrac{1}{\sqrt{2}}\right)\left[2\,E\left(\tfrac{1}{\sqrt{2}}\right) - K\left(\tfrac{1}{\sqrt{2}}\right)\right] &= \frac{\Gamma^2\left(\frac{1}{4}\right)}{4\sqrt{\pi}}\left[\frac{4\,\Gamma^2\left(\frac{3}{4}\right) + \Gamma^2\left(\frac{1}{4}\right)}{4\sqrt{\pi}} - \frac{\Gamma^2\left(\frac{1}{4}\right)}{4\sqrt{\pi}}\right] \\ &= \frac{\Gamma^2\left(\frac{1}{4}\right)}{4\sqrt{\pi}}\frac{\Gamma^2\left(\frac{3}{4}\right)}{\sqrt{\pi}} \\ &= \frac{1}{4\pi}\left[\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)\right]^2. \end{aligned}$$

Using Proposition 3.6(b) produces

$$K\left(\frac{1}{\sqrt{2}}\right)\left[2\,E\left(\frac{1}{\sqrt{2}}\right) - K\left(\frac{1}{\sqrt{2}}\right)\right] = \frac{1}{4\pi}\left[\sqrt{2}\,\pi\right]^2 = \frac{\pi}{2},$$

as required. □

The above corollary demonstrates a specific case of what is known as *Legendre's relation.* It states that

$$K(k)\,[2E(k) - K(k)] = \frac{\pi}{2}$$

is true for any $k \in (0,1)$, not just for $k = 1/\sqrt{2}$. The proof of this relation is in Borwein and Borwein [1].

5. An Algorithm for π

We now have all the necessary ingredients to produce an algorithm that computes π. This algorithm and the resulting corollaries are taken from Borwein and Borwein [1].

Algorithm 5.1. Let $a_0 = 1$ and $b_0 = 1/\sqrt{2}$. Define

$$\pi_n = \frac{2{a_{n+1}}^2}{1 - \sum_{k=0}^{n} 2^k {c_k}^2},$$

where ${c_n}^2 = {a_n}^2 - {b_n}^2$. Then, π_n increases monotonically to π.

Proof. Corollary 4.5 states

$$\frac{\pi}{2} = K(2E - K),$$

where $K = K(1/\sqrt{2})$ and $E = E(1/\sqrt{2})$. Then, using Theorem 4.3 gives

$$\begin{aligned}\frac{\pi}{2} &= K\left[2\left(1 - \sum_{n=0}^{\infty} 2^{n-1}{c_n}^2\right)K - K\right] \\ &= K^2\left[\left(2 - \sum_{n=0}^{\infty} 2^{n}{c_n}^2\right) - 1\right] \\ &= K^2\left[1 - \sum_{n=0}^{\infty} 2^{n}{c_n}^2\right].\end{aligned}$$

Proposition 2.4 tells us that

$$K = \frac{\pi}{2\,M\big(1, 1/\sqrt{2}\big)},$$

which allows us to write

$$\frac{\pi}{2} = \frac{\pi^2}{4\,M^2\big(1, 1/\sqrt{2}\big)}\left[1 - \sum_{n=0}^{\infty} 2^{n}{c_n}^2\right].$$

Rearranging gives

$$\pi = \frac{2\,M^2\big(1, 1/\sqrt{2}\big)}{1 - \sum_{n=0}^{\infty} 2^{n}{c_n}^2}. \tag{35}$$

If we truncate the infinite series at n and note that ${a_{n+1}}^2 \approx M^2\big(1, 1/\sqrt{2}\big)$ for large values of n, then we can write

$$\pi_n = \frac{2{a_{n+1}}^2}{1 - \sum_{k=0}^{n} 2^{k}{c_k}^2}.$$

Now we need to prove π_n increases monotonically to π. For clarity, denote the following summation by Σ^m:

$$\Sigma^m = \sum_{k=0}^{m} 2^{k}{c_k}^2.$$

Then, from Algorithm 5.1, we know that

$$\pi_n = \frac{2{a_{n+1}}^2}{1 - \Sigma^n} \qquad \text{and} \qquad \pi_{n+1} = \frac{2{a_{n+2}}^2}{1 - \Sigma^{n+1}}.$$

Now, take their difference to give

$$\begin{aligned}\pi_{n+1} - \pi_n &= \frac{2{a_{n+2}}^2}{1 - \Sigma^{n+1}} - \frac{2{a_{n+1}}^2}{1 - \Sigma^n} \\ &= \frac{2{a_{n+2}}^2\left(1 - \Sigma^n\right) - 2{a_{n+1}}^2\left(1 - \Sigma^{n+1}\right)}{\left(1 - \Sigma^{n+1}\right)\left(1 - \Sigma^n\right)}.\end{aligned} \tag{36}$$

Since $a_{n+1} \geqslant a_{n+2}$, we can write

$$\pi_{n+1} - \pi_n \geqslant \frac{2{a_{n+2}}^2\left(\Sigma^{n+1} - \Sigma^n\right)}{\left(1 - \Sigma^{n+1}\right)\left(1 - \Sigma^n\right)} = \frac{{a_{n+2}}^2\,2^{n+2}\,{c_{n+1}}^2}{\left(1 - \Sigma^{n+1}\right)\left(1 - \Sigma^n\right)}.$$

Now, since $(1-\Sigma^{n+1})(1-\Sigma^n) \leqslant (1-\Sigma^n)^2$, we can write

$$\pi_{n+1} - \pi_n \geqslant \frac{{a_{n+2}}^2\, 2^{n+2}\, {c_{n+1}}^2}{(1-\Sigma^n)^2}.$$

Then, it is trivial to see that $\pi_{n+1} - \pi_n \geqslant 0$ and therefore that π_n increases monotonically. From (35), we can see that π_n increases to π. ☐

The following corollary gives an upper bound for $\pi_{n+1} - \pi_n$, that is the difference between two consecutive iterations of the algorithm.

Corollary 5.2. *In Algorithm 5.1,*

$$\pi_{n+1} - \pi_n \leqslant \frac{2^n\, {c_{n+1}}^2\, \pi^2}{M^2\big(1, 1/\sqrt{2}\big)}.$$

Proof. Equation (36) from the proof above states

$$\pi_{n+1} - \pi_n = \frac{2{a_{n+2}}^2\,(1-\Sigma^n) - 2{a_{n+1}}^2\left(1-\Sigma^{n+1}\right)}{(1-\Sigma^{n+1})\,(1-\Sigma^n)}.$$

Now, since $(1-\Sigma^{n+1})(1-\Sigma^n) \geqslant (1-\Sigma^\infty)^2$, we can write

$$\pi_{n+1} - \pi_n \leqslant \frac{2{a_{n+2}}^2\,(1-\Sigma^n) - 2{a_{n+1}}^2\left(1-\Sigma^{n+1}\right)}{(1-\Sigma^\infty)^2}.$$

Then, since $a_{n+2} \leqslant a_{n+1}$,

$$\begin{aligned}\pi_{n+1} - \pi_n &\leqslant \frac{2{a_{n+1}}^2\left[(1-\Sigma^n) - \left(1-\Sigma^{n+1}\right)\right]}{(1-\Sigma^\infty)^2}\\ &= \frac{2{a_{n+1}}^2\left[\Sigma^{n+1} - \Sigma^n\right]}{(1-\Sigma^\infty)^2}.\end{aligned}$$

Observing that $\Sigma^{n+1} = \Sigma^n + 2^{n+1}\,{c_{n+1}}^2$, we can write

$$\pi_{n+1} - \pi_n \leqslant \frac{2{a_{n+1}}^2\, 2^{n+1}\, {c_{n+1}}^2}{(1-\Sigma^\infty)^2}.$$

From (35), we have

$$\pi = \frac{2\,M^2\big(1, 1/\sqrt{2}\big)}{(1-\Sigma^\infty)},$$

which rearranges to

$$(1-\Sigma^\infty)^2 = \frac{4\,M^4\big(1, 1/\sqrt{2}\big)}{\pi^2}.$$

Substituting this leads to

$$\pi_{n+1} - \pi_n \leqslant \frac{2{a_{n+1}}^2\, 2^{n+1}\, {c_{n+1}}^2\, \pi^2}{4\,M^4\big(1, 1/\sqrt{2}\big)} = \frac{{a_{n+1}}^2\, 2^n\, {c_{n+1}}^2\, \pi^2}{M^4\big(1, 1/\sqrt{2}\big)}.$$

In the proof of Algorithm 5.1, we approximated $M^2\big(1,1/\sqrt{2}\big)$ by ${a_{n+1}}^2$ and we shall do the same here. This gives

$$\pi_{n+1} - \pi_n \leqslant \frac{M^2\big(1,1/\sqrt{2}\big)\ 2^n\, {c_{n+1}}^2\, \pi^2}{M^4\big(1,1/\sqrt{2}\big)} = \frac{2^n\, {c_{n+1}}^2\, \pi^2}{M^2\big(1,1/\sqrt{2}\big)},$$

as required. □

This corollary gives an upper bound for $\pi - \pi_n$, that is the difference between an iteration and the true value of π. It can be used to calculate the number of correct digits in π_n — in fact, the script in the next section uses this corollary to do exactly that.

Corollary 5.3. *In Algorithm 5.1,*

$$\pi - \pi_n \leqslant \frac{\pi^2\, 2^{n+4}\, \exp\big\{-\pi 2^{n+1}\big\}}{M^2\big(1,1/\sqrt{2}\big)}.$$

This proof assumes the following statement:

$$\lim_{n\to\infty} 2^{-n} \log\left(\frac{4a_n}{c_n}\right) = \frac{\pi}{2}\,\frac{M(1,k')}{M(1,k)}. \tag{37}$$

The details of which are covered in Borwein and Borwein [1]. These details make use of Jacobi's *theta* functions — which is beyond on the scope of this project.

Proof. From (37), we can say

$$\lim_{n\to\infty} 2^{1-n} \log\left(\frac{4a_n}{c_n}\right) = \pi$$

because $M(1,k') = M(1,k)$ when $k = k' = 1/\sqrt{2}$. Then, further rearranging produces

$$\lim_{n\to\infty} \left(\frac{4a_n}{c_n}\right)^{2^{1-n}} = e^{\pi},$$

and taking reciprocals gives

$$\lim_{n\to\infty} \left(\frac{c_n}{4a_n}\right)^{2^{n-1}} = e^{-\pi}.$$

By substituting $n+1$ for n, we can write

$$\lim_{n\to\infty} \left(\frac{c_{n+1}}{4a_{n+1}}\right)^{2^{-n}} = e^{-\pi}.$$

Dividing through by $e^{-\pi}$ gives

$$\lim_{n\to\infty} \left(\frac{c_{n+1}}{4a_{n+1}}\right)^{2^{-n}} \Bigg/ e^{-\pi} = 1,$$

and, by raising to the power of 2^{n+1},

$$\lim_{n\to\infty} \left(\frac{c_{n+1}}{4a_{n+1}}\right)^{2} \Bigg/ e^{-\pi\, 2^{n+1}} = 1.$$

Then, for large enough n, we can say

$$\frac{{c_{n+1}}^2}{4{a_{n+1}}^2} \bigg/ e^{-\pi\, 2^{n+1}} \leqslant 1,$$

which, because each variable is positive, rearranges to

$${c_{n+1}}^2 \leqslant 16{a_{n+1}}^2\, e^{-\pi\, 2^{n+1}}.$$

Using the fact that $16{a_{n+1}}^2 \leqslant 16{a_1}^2$, we can say that $16{a_{n+1}}^2 \leqslant 12$ — which we can say because $16{a_1}^2 = 6 + 4\sqrt{2} \approx 11.66$. Therefore,

$${c_{n+1}}^2 \leqslant 12e^{-\pi\, 2^{n+1}}. \tag{38}$$

Now, because π_n increases monotonically and the AGM converges quadratically (see Corollary 1.4), we can write

$$\pi_{j+2} - \pi_{j+1} \leqslant \tfrac{1}{4}\left(\pi_{j+1} - \pi_j\right),$$

for some non-negative integer j. Hence, we can say

$$\begin{aligned}
\pi_{n+2} - \pi_n &\leqslant \tfrac{5}{4}\left(\pi_{n+1} - \pi_n\right),\\
\pi_{n+3} - \pi_n &\leqslant \tfrac{21}{16}\left(\pi_{n+1} - \pi_n\right),\\
\pi_{n+4} - \pi_n &\leqslant \tfrac{85}{64}\left(\pi_{n+1} - \pi_n\right),\\
&\vdots\\
\pi - \pi_n &\leqslant \tfrac{4}{3}\left(\pi_{n+1} - \pi_n\right).
\end{aligned} \tag{39}$$

Corollary 5.2 states that

$$\pi_{n+1} - \pi_n \leqslant \frac{2^n\, {c_{n+1}}^2\, \pi^2}{M^2\big(1, 1/\sqrt{2}\big)},$$

and combining with (39) produces

$$\pi - \pi_n \leqslant \frac{4}{3}\, \frac{2^n\, {c_{n+1}}^2\, \pi^2}{M^2\big(1, 1/\sqrt{2}\big)}.$$

Finally, using (38) with the above gives

$$\pi - \pi_n \leqslant 12\, \frac{4}{3}\, \frac{2^n\, {c_{n+1}}^2\, \pi^2}{M^2\big(1, 1/\sqrt{2}\big)} = \frac{2^{n+4}\, {c_{n+1}}^2\, \pi^2}{M^2\big(1, 1/\sqrt{2}\big)},$$

as required. □

As Example 6.2 shows in the next section, the first four iterations of the algorithm produce the following values:

n	π_n
0	2.91
1	3.14
2	3.1415926
3	3.14159265358979323 8

Since the algorithm converges quadratically, the correct number of digits increases very quickly. Shown below are the first ten iterations of the algorithm and the number of correct digits they produce:

iteration	0	1	2	3	4	5	6	7	8	9
digits	0	3	8	19	41	84	171	345	694	1392

Although this project covers just one algorithm that uses the AGM, there are many more included in Borwein and Borwein [1]. Some of those covered converge much quicker than ours, for instance, one has septic convergence (meaning the number of correct digits multiplies by 7 each iteration).

To conclude, in this project, we have progressed from the origins of the arithmetic-geometric mean to forming an algorithm for π. The understanding of elliptic integrals and their relationship to both the AGM and the gamma function was key to that progression. If I had more time, I would like to look further into the algorithms referred to above as well an variation of the AGM for complex numbers. In the next section I will discuss how I applied the mathematics that I have learnt throughout this project in the form of a computer script.

6. Computation

To support this project, I have written a script in the computer language Python. The first function of the script calculates the AGM in the manner described in Section 1 and the second function calculates π using Algorithm 5.1, as described in the previous section. The two examples below demonstrate this functionality. (The script's source code is displayed in the Appendix.)

Example 6.1. In Example 1.2, we calculated that the AGM of 25 and 4 is approximately 12.146. We can calculate this more precisely using the script. We give the script five arguments:

- `agm`, which selects the AGM calculation function;
- `-p15`, which sets the precision to 15 significant figures;
- `-v`, which turns verbosity on to print each iteration;
- `25`, which is our value for a;
- `4`, which is our value for b.

The command and its arguments are in bold font with the output displayed below it. Note that `i=n` refers to the number of the current iteration.

```
./agm.py agm -p15 -v 25 4
-> i=0                              -> i=1
        a: 25                               a: 14.5
        b: 4                                b: 10
-> i=2                              -> i=3
        a: 12.25                            a: 12.1457972893961
        b: 12.0415945787923                 b: 12.1453502868466
-> i=4                              -> i=5
        a: 12.1455737881214                 a: 12.1455737870932
        b: 12.1455737860650                 b: 12.1455737870932

  agm: 12.1455737870932
 prec: 15
```

As can be seen, the script required 5 iterations to reach the desired number of significant figures (denoted by `prec` in the output).

The script can be run with any positive integer for the precision and shown below is the AGM of the same two values calculated to 175 significant figures (the `-q` option means that only the result will be printed).

```
./agm.py agm -p175 -q 25 4
12.14557378709318059673123191493610156748726895906910273800 9
  632800827123965266981421192912045179035907017918575369639 0
  4793061298881553605552740386979811591631271353663828178629
```

Example 6.2. To calculate π, we give the script three arguments:

- `pi`, which selects the π calculation function;
- `-v`, which turns verbosity on like before;
- `3`, which is our value for n — the number of iterations to perform.

As before, the command is in bold font, the script's output is displayed below it and `i=n` is the current iteration.

```
./agm.py pi -v 3
-> i=0                          -> i=1
        pi: 2.91                        pi: 3.14
      prec: 0                         prec: 3
-> i=2                          -> i=3
        pi: 3.1415926                   pi: 3.141592653589793238
      prec: 8                         prec: 19

   pi: 3.141592653589793238
 prec: 19
```

Similarly, we can run many more iterations to produce a value for π with a higher precision. Here, there are seven iterations which produces 345 digits.

```
./agm.py pi 7
-> i=7
   pi: 3.1415926535897932384626433832795028841971693993751 05
        820974944592307816406286208998628034825342117067982 1
        480865132823066470938446095505822317253594081284811 1
        745028410270193852110555964462294895493038196442881 0
        975665933446128475648233786783165271201909145648566 9
        234603486104543266482133936072602491412737245870066 0
        63155881748815209209628292540917 1
 prec: 345
```

Technical Details. The script makes use of Python's *decimal* module which allows for calculations requiring *any* number of significant figures. Be aware that some calculations involving very high precisions could require a long time to complete. For example, ten iterations of calculating π took about 21 seconds to produce 2789 digits whereas eleven iterations took about 154 seconds to produce 5583 digits — this is roughly twice as many digits but over seven times the time taken.

References

[1] J. M. Borwein and P. B. Borwein, *Pi and the AGM*, (Wiley-Interscience, 1987), 1–174.
[2] D. A. Cox, "The Arithmetic-Geometric Mean of Gauss," *L'Enseignement Mathématique.* **30** (1984), 275–330.
[3] C. F. Gauss, *Werke*, (Göttingen, 1876), 352–353.
[4] J. Havil, *Gamma: Exploring Euler's Constant*, (Princeton University Press, 2003), 47–59.
[5] E. C. Titchmarsh, *The Theory of Functions*, Second Edition, (Oxford University Press, 1939), 55–56.

Appendix

Included here is the latest version of the script. (Latest means at time of LaTeX compilation — December 22, 2014 in this case.) Long lines are broken, where the '↪' symbol indicates a break. A later version of the script may be available at `https://bitbucket.org/rowanparkeruk/agm` and the author can be reached at `rowan@rowanparker.com`.

```
# agm.py - Computes the arithmetic-geometric mean at any precision and uses it
#          to calculate pi.
# Copyright (c) 2013-14 Rowan Parker (rowan at rowanparker dot com)
#
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# SOFTWARE.

from __future__ import print_function
from decimal import *
import argparse
import sys

class AGM(object):
    def __init__(self, precision=30, print_agm=True, print_errors=True, print_counts=True,
     ↪ print_steps=False, pi_mode=False):
        if int(precision) < 1:
            self.prec = 30
        else:
            self.prec = int(precision)
        getcontext().rounding = ROUND_HALF_UP
        getcontext().prec = self.prec + 2
        self.print_agm = bool(print_agm)
        self.print_errors = bool(print_errors)
        self.print_steps = bool(print_steps)
        self.print_counts = bool(print_counts)
        self.pi_mode = bool(pi_mode)
    def change_prec(self, precision):
        if int(precision) > 2:
            self.prec = int(precision)
        else:
            if self.print_errors:
                print("FAIL: %s.change_prec(%s) returned false because the new precision was lower than
                     ↪ 2." % (self.__class__.__name__, precision))
```

```
            return False
        getcontext().prec = self.prec + 2
        return True
    def decimal_validate(self, val):
        try:
            return Decimal(val)
        except InvalidOperation:
            a = val.split('^')
            if len(a) == 2:
                t = [self.decimal_validate(a[0]), self.decimal_validate(a[1])]
                if False in t:
                    return False
                else:
                    return t[0]**t[1]
            b = val.split('/')
            if len(b) == 2:
                t = [self.decimal_validate(b[0]), self.decimal_validate(b[1])]
                if False in t:
                    return False
                else:
                    return t[0]/t[1]
            return False
    def start(self, ai, bi):
        if self.print_steps or self.print_counts:
            print("->agm: started", end="\r")
        a = self.decimal_validate(ai)
        b = self.decimal_validate(bi)
        if a > 0 and b > 0:
            if a > b:
                self.a = [a]
                self.b = [b]
            else:
                self.a = [b]
                self.b = [a]
            if self.pi_mode:
                self.csum = [getcontext().power(self.a[-1],2) - getcontext().power(self.b[-1],2)]
            with localcontext() as ctx:
                ctx.prec = self.prec
                self.ap = [+self.a[-1]]
                self.bp = [+self.b[-1]]
            self.finished = False
            self.iterate_print()
            return True
        else:
            if self.print_errors:
                print("FAIL: %s.start(%s, %s) returned false because of an error with a and/or b." % (self.__class__.__name__, ai, bi))
            return False
    def iterate(self):
        if self.finished:
            return False
        if not self.a or not self.b:
            if self.print_errors:
                print("FAIL: %s.iterate() returned false because %s.a and/or %s.b were not set. Have you run %s.start() first?" % (self.__class__.__name__, self.__class__.__name__, self.__class__.__name__))
            return False
        am = (self.a[-1]+self.b[-1])/Decimal(2)
        gm = (self.a[-1]*self.b[-1]).sqrt()
        self.a.append(am)
        self.b.append(gm)
        if self.pi_mode:
            t2k = getcontext().power(2, len(self.a)-1)
            tcsq = (getcontext().power(self.a[-1],2) - getcontext().power(self.b[-1],2))
            self.csum.append(self.csum[-1]+t2k*tcsq)
        with localcontext() as ctx:
            ctx.prec = self.prec
            self.ap.append(+self.a[-1])
            self.bp.append(+self.b[-1])
        self.iterate_print()
        if (self.a[-1] - self.b[-1]).adjusted() <= -self.prec:
            self.finished = True
        return True
    def iterate_print(self):
        if self.print_counts or self.print_steps:
            print("-> i=%i" % (len(self.ap)-1), end=" "*30 + "\r")
        if self.print_steps:
            print("\n\t a: %s\n\t b: %s" % (self.ap[-1], self.bp[-1]))
```

```
            if self.pi_mode:
                print("\t\b\bcsum: %s" % self.csum[-1])
    def answer(self):
        if self.finished:
            if self.print_agm:
                if self.print_counts or self.print_steps:
                    print("\n  agm: ", end="")
                print(self.ap[-1])
                if self.print_steps:
                    print(" prec: " + str(self.prec))
            return self.ap[-1]
        else:
            if self.print_errors:
                print("FAIL: %s.answer() returned false because %s.finished is false (the algorithm has not finished)." % (self.__class__.__name__, self.__class__.__name__))
            return False
    def calculate(self, a, b):
        if not self.start(a, b):
            return False
        while not self.iterate():
            pass
        return self.answer()

class Pi(object):
    def __init__(self, print_pi=True, print_errors=True, print_counts=True, print_steps=False, print_agm_steps=False):
        self.print_pi = bool(print_pi)
        self.print_errors = bool(print_errors)
        self.print_steps = bool(print_steps)
        self.print_counts = bool(print_counts)
        self.print_agm_steps = bool(print_agm_steps)
        self.agm = AGM(3, False, self.print_errors, self.print_counts, self.print_agm_steps, True)
    def digits(self, n, pi=False, M=False):
        if n % 1 != 0 or n < 0:
            return False
        if n == 0:
            return 0
        if not pi:
            pi = Decimal('3.142')
        if not M:
            M = Decimal('0.8472')
        a = 2*(getcontext().ln(pi)-getcontext().ln(M))
        b = (n+4)*getcontext().ln(2)
        c = pi*getcontext().power(2, n+1)
        d = (a+b-c)/getcontext().ln(10)
        return int(-d.to_integral_exact(rounding = ROUND_FLOOR))
    def start(self, n):
        if int(n) < 0:
            if self.print_errors:
                print("FAIL: %s.start(%s) returned false because n was not positive." % (self.__class__.__name__, n))
            return False
        else:
            self.n = int(n)
        guess_digits = self.digits(self.n)
        if guess_digits < 3:
            guess_digits += 3
        getcontext().prec = guess_digits+2
        getcontext().rounding = ROUND_HALF_UP
        if not self.agm.change_prec(guess_digits+2):
            return False
        return self.agm.start(Decimal('1'), (Decimal('1')/Decimal('2')).sqrt())
    def iterate(self):
        self.agm.iterate()
        self.iterate_print()
    def iterate_print(self):
        if self.print_steps:
            if not self.print_agm_steps:
                print()
            print("\tpi: " + str(self.equation()))
            print("\t\b\bprec: " + str(self.correct_digits))
    def equation(self):
        if not self.agm.csum:
            return False
        tn = getcontext().power(self.agm.a[-1] + self.agm.b[-1], 2)/Decimal('2')
        td = 1 - self.agm.csum[-1]
        pi = tn/td
        self.correct_digits = self.digits(self.i, pi, self.agm.a[-1])
```

```python
        with localcontext() as ctx:
            if self.correct_digits < 3:
                ctx.prec = 3
            else:
                ctx.prec = self.correct_digits
            return +pi
    def answer(self):
        if self.i == self.n:
            pi = self.equation()
            if self.print_pi:
                if self.print_counts or self.print_steps:
                    print("\n   pi: ", end="")
                print(pi)
                if self.print_counts or self.print_steps:
                    print(" prec: " + str(self.correct_digits))
            return pi
        else:
            if self.print_errors:
                print("FAIL: %s.answer() returned false because %s.i != %s.n (the algorithm has not finished)." % (self.__class__.__name__, self.__class__.__name__, self.__class__.__name__))
            return False
    def calculate(self, n):
        if not self.start(n):
            return False
        self.i = 0
        self.iterate_print()
        for i in range(1, n+1):
            self.i = i
            self.iterate()
        return self.answer()

def main():
    parser = argparse.ArgumentParser(description="A script written to calculate the arithmetic-geometric mean to an arbitrary precision, and use it to calculate pi.")
    subparsers = parser.add_subparsers(help="the script's function")
    agm_parser = subparsers.add_parser('agm', help="calculate the arithmetic-geometic mean - use `agm -h` for more info")
    agm_parser.add_argument("a", help="the first value - use / (a forward slash) for fractions and ^ (a caret) for exponentials")
    agm_parser.add_argument("b", help="the second value - same as first")
    agm_parser.add_argument("-p", "--precision", help="change the precision to the positive integer P from the default value of 30", type=int, default=30, metavar='P')
    agm_parser.add_argument("-v", "--verbose", help="print each step of the agm calculation", action="store_true")
    agm_parser.add_argument("-q", "--quiet", help="surpress all output other than the result (and errors)", action="store_true")
    agm_parser.set_defaults(func=do_agm)
    pi_parser = subparsers.add_parser('pi', help="calculate pi - use `pi -h` for more info")
    pi_parser.add_argument("n", help="the number of iterations to perform (the first iteration is n=0)", type=int)
    pi_parser.add_argument("-v", "--verbose", help="print each step of the pi calculation", action="store_true")
    pi_parser.add_argument("-vv", "--veryverbose", help="print each step of the pi and agm calculation", action="store_true")
    pi_parser.add_argument("-q", "--quiet", help="surpress all output other than the result (and errors)", action="store_true")
    pi_parser.set_defaults(func=do_pi)
    args = parser.parse_args()
    args.func(args)

def do_agm(args):
    if args.quiet:
        args.print_counts = False
        args.verbose = False
    else:
        args.print_counts = True
    agm = AGM(args.precision, True, True, args.print_counts, args.verbose, False)
    agm.calculate(args.a, args.b)

def do_pi(args):
    if args.veryverbose:
        args.verbose = True
    if args.quiet:
        args.print_counts = False
        args.verbose = False
        args.veryverbose = False
    else:
```

```
        args.print_counts = True
    pi = Pi(True, True, args.print_counts, args.verbose, args.veryverbose)
    pi.calculate(args.n)

if __name__ == '__main__':
    try:
        main()
    except KeyboardInterrupt:
        print("\nInterrupted.")
        exit()
```